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The generation of sound by aerodynamic sources in an inhomogeneous steady flow

By M. S. HOWE
Engineering Department, University of Cambridge

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This paper examines the effect of steady flow on the generation of sound by aerodynamic sources located in the neighbourhood of a scattering body. The analysis is facilitated by the use of a reverse-flow reciprocal theorem which is valid for a certain class of inhomogeneous flow problems. In the case of acoustic 'bremsstrahlung', in which sound is generated during the passage of a *silent* fluid inhomogeneity, such as an entropy spot, past an obstacle in the flow, an elegant discussion of the interaction can be given in terms of a *renormalized* Green's function. Here the effect of the obstacle is equivalent to a local distortion of the geometry of the three-dimensional space occupied by the medium, the extent of which governs the properties of the acoustic radiation. This is illustrated by means of a detailed analysis of the sound generated during the unsteady convection of a source of constant strength through a neck in a duct.

1. Introduction

The Lighthill (1952, 1954) theory of aerodynamic sound is based on an acoustic analogy in which turbulence provides a quadrupole source distribution in an ideal fluid at rest. In the development of the theory Lighthill, and later Ffowcs Williams (1963) and Ffowcs Williams & Hawkings (1969) emphasized the need to take account of source-convection effects in estimating the radiated sound pressure levels. More recently Ffowcs Williams (1974), in a study of noise generation by a turbulent shear layer, has demonstrated the desirability of expressing the temporal variations of source distributions in motion in terms of material derivatives following that motion. This is particularly important in the case of a source such as, for example, an *entropy spot,* which is acoustically silent when convecting uniformly, but emits sound (acoustic 'bremsstrahlung') when it encounters a flow inhomogeneity such as would be caused by the presence of an obstacle in the fluid.

In the absence of flow the analysis of the acoustic radiation from a source in the neighbourhood of a scattering body is often facilitated by use of the *reciprocal theorem.* The Helmholtz (1860) reciprocal theorem states that, in the presence of arbitrary, fixed rigid scattering surfaces, the acoustic velocity potential at a point B due to a point source of sound at A is the same as it would have been at A had the source been situated at B. In a discussion of more general systems, Rayleigh (1873) showed that the theorem remains true in the presence of dissipative forces, provided that these arise from resistances proportional to the first power of the velocity, and that the fluid need not be homogeneous, nor the scattering surfaces rigid. Rayleigh (1876) also clarified the correct interpretation of the theorem when the acoustic source has a dipole or quadrupole character rather than that of a simple monopole.

The theorem has been applied to the study of the enhancement of aerodynamic noise from sources located in the neighbourhood of sharp scattering surfaces. The presence of a scattering edge converts near-field, essentially incompressible turbulent flow fluctuations into radiating sound waves (Ffowcs Williams & Hall 1970). Mention may be made, in particular, of the analysis of Crighton & Leppington (1970), who used the theorem to examine the scattering of sound from quadrupole sources located near the edge of a semi-infinite *compliant* plate. Crighton & Leppington (1973) have also considered the problem of the scattering of sound by a thick rigid plate, and invoked the reciprocal theorem to justify their procedure for the matching of asymptotic expansions valid in different parts of the wave field.

The author is not aware of any attempts to apply similar methods to problems in which the mean state of the fluid medium is one of steady flow, and in which acoustic sources are convected past fixed scattering bodies. Such flow regimes represent a more realistic approximation to the environment of noise sources located in engine ducts and near struts and trailing edges. In this paper an approach is made towards these more general problems by means of a simple extension of the reciprocal theorem. This extension constitutes a 'reverse flow' theorem wherein, in the reciprocal problem, in which source and observer are interchanged, the steady mean flow is *reversed* at all points in space. Such reverse-flow theorems are familiar in lifting-surface theory (see, for example, Flax 1952; Miles 1959; Ogilvie 1973), although in that theory the lifting surfaces are required to assume a linearized plan-form, so that the mean reversible flow velocity is uniform at all points in space. On the contrary we shall be concerned with situations in which the steady perturbation of the uniform stream produced by the presence of a scattering body is *nonlinear*, so that the acoustic problem is one of sound propagation in a medium in which the steady flow is inhomogeneous. Naturally such a theorem is valid only for a rather restricted class of steady flow regimes. In particular it is necessary to assume that $M^2 \ll 1$, where M is a characteristic Mach number of the flow. If the acoustic wavelengths involved are *not* small compared with the scale of variation of the mean velocity field it is also necessary to assume that the mean velocity field is *irrotational*.

From a practical point of view these restrictions are severe. Nevertheless the theorem permits the solution with relative ease of several apparently difficult model problems. The analysis of such problems, it can be argued, must inevitably lead to a greater degree of insight into the mechanisms of scattering and diffraction in real inhomogeneous flows. Under the same restriction on Mach number, the principal acoustic properties of these flows would hardly be expected to differ substantially from predictions based on the appropriate model.

The reverse-flow reciprocal theorem is discussed in §2 of this paper. In §3 the theorem is used to determine the sound radiated by a point source located in the vicinity of a rigid circular cylinder in the presence of a steady streaming flow. This is relevant to the study of the effect of aircraft flight on the radiation from aerodynamic noise sources located near scattering surfaces. The analysis of problems involving source motion relative to the scattering body is slightly more complicated, but in problems of acoustic bremsstrahlung, in which sound is generated only during the passage of an acoustically silent source past a body, an elegant analysis can be conducted in terms of a renormalized Green's function (§4). The functional form of this Green's function is identical with that appropriate to uniform flow in free space, but in the neighbourhood of the scattering body the effect of that body is equivalent to a local distortion of the three-dimensional Euclidean space occupied by the medium, the extent of which governs the properties of the acoustic bremsstrahlung. This is illustrated in §5 by a discussion of the problem of the sound generated when a quiescent point source (i.e. a source of *constant* strength) is convected in a low Mach number steady flow through a neck in a duct. When the source is far from the neck conditions are essentially steady and no sound is generated. The unsteady motion of the source in passing through the neck, however, generates a compression wave which passes downstream ahead of the source and an expansion wave which radiates upstream from the neck.

2. Reverse-flow reciprocity

Consider a configuration in which fluid extending to infinity contains a system of closed surfaces S. We shall assume for the purposes of this discussion that the surfaces are of finite extent, although the more general case involving surfaces extending to infinity requires only a slight modification of the following argument. The surfaces are fixed in space, and the fluid is assumed to be inviscid and in *steady incompressible* motion with velocity $\mathbf{U}(\mathbf{x})$, i.e. such that

$$\operatorname{div}\mathbf{U} \equiv \partial U_i/\partial x_i = 0. \tag{2.1}$$

In the steady flow the normal velocity on S must vanish. This is the case even if the surfaces S are compliant or can themselves support small mechanical vibrations.

Let ϕ denote the velocity potential of small disturbances in the fluid. On S we assume that ϕ satisfies an impedance condition of the form

$$l_j\,\partial\phi/\partial x_j = \lambda\phi, \tag{2.2}$$

where $\mathbf{l}$ is a unit normal to S, and where in general λ depends on the properties of S *and* on the local mean flow velocity $\mathbf{U}$, i.e. $\lambda \equiv \lambda(\mathbf{U})$.

We now restrict our attention to situations in which ϕ satisfies the convected wave equation

$$(\partial/\partial t + U_j\,\partial/\partial x_j)^2\,\phi - c^2\nabla^2\phi = 0, \tag{2.3}$$

where the speed of sound c is assumed to be constant. We shall discuss below the precise conditions under which (2.3) is appropriate to the practical problem of acoustics.

Now consider a harmonic point source

$$q = e^{-i\omega t}\,\delta(\mathbf{x}-\mathbf{x}_A), \tag{2.4}$$

located at $\mathbf{x} = \mathbf{x}_A$. Then the time-reduced equation for the potential ϕ_A, say, is

$$(-i\omega + U_j \partial/\partial x_j)^2 \phi_A - c^2\nabla^2\phi_A = \delta(\mathbf{x}-\mathbf{x}_A), \tag{2.5a}$$

with

$$l_j\, \partial\phi_A/\partial x_j = \lambda(\mathbf{U})\,\phi_A \quad \text{on} \quad S. \tag{2.5b}$$

Next consider the reciprocal problem for a point source

$$e^{-i\omega t}\delta(\mathbf{x}-\mathbf{x}_B)$$

in which the reduced potential ϕ_B satisfies the reverse-flow convected wave equation

$$(-i\omega - U_j \partial/\partial x_j)^2 \phi_B - c^2\nabla^2\phi_B = \delta(\mathbf{x}-\mathbf{x}_B), \tag{2.6a}$$

with

$$l_j \partial\phi_B/\partial x_j = \lambda(\mathbf{U})\,\phi_B \quad \text{on} \quad S. \tag{2.6b}$$

Then the reverse-flow reciprocal theorem asserts that

$$\phi_A(\mathbf{x}_B) = \phi_B(\mathbf{x}_A). \tag{2.7}$$

Note that in (2.6) the mean convection velocity $\mathbf{U}$ has been reversed at all points of the flow in the wave equation, but *not* in the boundary condition on S. Observe also that the reversed flow does not necessarily correspond to a possible steady flow of the real system. The reversed flow is possible, of course, if $\mathbf{U}$ is an *irrotational* velocity field.

The proof of (2.7) is straightforward, and the details differ from the usual proof (see, for example, Jones 1964, p. 63) only because of the presence of the terms in $\mathbf{U}$ in (2.5) and (2.6). The procedure consists of multiplying (2.5) by ϕ_B and (2.6) by ϕ_A, subtracting the two equations and integrating over the volume V contained between a large spherical control surface Σ and the surfaces S. This gives

$$\begin{aligned}\phi_B(\mathbf{x}_A)-\phi_A(\mathbf{x}_B) = &-2i\omega\int_V \left[\phi_B U_j \frac{\partial\phi_A}{\partial x_j} + \phi_A U_j \frac{\partial\phi_B}{\partial x_j}\right] d^3\mathbf{x} \\ &+\int_V \left[\phi_B U_i \frac{\partial}{\partial x_i}\left(U_j \frac{\partial\phi_A}{\partial x_j}\right) - \phi_A U_i \frac{\partial}{\partial x_i}\left(U_j \frac{\partial\phi_B}{\partial x_j}\right)\right] d^3\mathbf{x} \\ &-c^2\int_V [\phi_B\nabla^2\phi_A - \phi_A\nabla^2\phi_B]\, d^3\mathbf{x}. \end{aligned} \tag{2.8}$$

Using the divergence theorem, and noting that $\partial U_i/\partial x_i = 0$, (2.8) becomes

$$\begin{aligned}\phi_B(\mathbf{x}_A)-\phi_A(\mathbf{x}_B) = &-2i\omega\oint_{S+\Sigma} l_j U_j \phi_A \phi_B\, d^2\mathbf{x} \\ &+\oint_{S+\Sigma} l_i U_i U_j \left(\phi_B \frac{\partial\phi_A}{\partial x_j} - \phi_A \frac{\partial\phi_B}{\partial x_j}\right) d^2\mathbf{x} \\ &-c^2\oint_{S+\Sigma} l_j \left(\phi_B \frac{\partial\phi_A}{\partial x_j} - \phi_A \frac{\partial\phi_B}{\partial x_j}\right) d^2\mathbf{x}.\end{aligned}$$

The contribution to the surface integral from S vanishes on using (2.5b) and (2.6b) and recalling that $l_j U_j \equiv 0$ on S. The contribution from Σ can also be seen to vanish on noting that the asymptotic forms for ϕ_A and ϕ_B are respectively

$$\phi_A \sim \frac{1}{4\pi c^2|\mathbf{x}|} F_A\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \exp\left\{-i\omega\left[t + \frac{\mathbf{M}_0.\mathbf{x}}{c(1-M_0^2)} - \frac{1}{c}\left(\frac{|\mathbf{x}|^2}{1-M_0^2} + \frac{(\mathbf{M}_0.\mathbf{x})^2}{(1-M_0^2)^2}\right)^{\frac{1}{2}}\right]\right\},$$

$$\phi_B \sim \frac{1}{4\pi c^2|\mathbf{x}|} F_B\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \exp\left\{-i\omega\left[t - \frac{\mathbf{M}_0.\mathbf{x}}{c(1-M_0^2)} - \frac{1}{c}\left(\frac{|\mathbf{x}|^2}{1-M_0^2} + \frac{(\mathbf{M}_0.\mathbf{x})^2}{(1-M_0^2)^2}\right)^{\frac{1}{2}}\right]\right\},$$

where $F_A(\mathbf{x}/|\mathbf{x}|)$ and $F_B(\mathbf{x}/|\mathbf{x}|)$ are functions of the radiation direction $\mathbf{x}/|\mathbf{x}|$ alone, and $\mathbf{M}_0$ is the Mach number of the asymptotic uniform flow of the medium. This proves (2.7).

Let us now examine the conditions under which (2.3) is likely to be an adequate characterization of the potential of a real flow.

The steady mean flow at velocity $\mathbf{U}$ may be regarded as incompressible provided that $M^2 \ll 1$, where $\mathbf{M} = \mathbf{U}/c$ (see, for example, Batchelor 1970, p. 167). Since variations in the speed of sound c also depend on the square of the mean flow Mach number, this condition also ensures that c^2 may be assumed to be constant in (2.3).

In the disturbed flow the velocity $\mathbf{V}$ is given by

$$\mathbf{V} = \mathbf{U} + \nabla\phi, \tag{2.9}$$

if it is assumed that the disturbance of the flow from the steady state is irrotational. The linearized momentum equation then has the form

$$\frac{\partial^2\phi}{\partial t\,\partial x_i} + U_j\frac{\partial^2\phi}{\partial x_j\,\partial x_i} + \frac{\partial U_i}{\partial x_j}\frac{\partial\phi}{\partial x_j} + \frac{c^2}{\rho_0}\frac{\partial\rho}{\partial x_i} = 0, \tag{2.10}$$

where, as above, the fluid is assumed to be inviscid, and where ρ_0 is the constant, undisturbed fluid density and ρ is the perturbation density.

We now restrict our considerations to two possible cases. The first is that in which the acoustic wavelength is much smaller than the length scale associated with the variation of $\mathbf{U}$, i.e.

$$\frac{1}{U}\frac{\partial U}{\partial x_i} \ll \frac{1}{\phi}\frac{\partial\phi}{\partial x_i}.$$

Then, since the speed of sound is effectively constant when $M^2 \ll 1$, equation (2.10) becomes approximately

$$\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j}\right)\phi + \frac{c^2\rho}{\rho_0} = 0. \tag{2.11}$$

Second, (2.11) is also valid for *arbitrary* wavelength if $\mathbf{U}$ is *irrotational*. For in that case $\partial U_i/\partial x_j \equiv \partial U_j/\partial x_i$, and under the same condition on c^2 equation (2.10) may be integrated exactly to yield (2.11).

The linearized continuity equation has the form

$$\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j}\right)\frac{\rho}{\rho_0} + \nabla^2\phi = 0, \tag{2.12}$$

so that the elimination of ρ/ρ_0 between (2.11) and (2.12) leads directly to the convected wave equation (2.3).

Note that the perturbation pressure, which is the parameter of physical importance in acoustics, is given in terms of ϕ by

$$\frac{p}{\rho_0} = -\frac{\partial\phi}{\partial t} - U_j\frac{\partial\phi}{\partial x_j}, \tag{2.13}$$

and does *not* satisfy an equation as simple as (2.3).

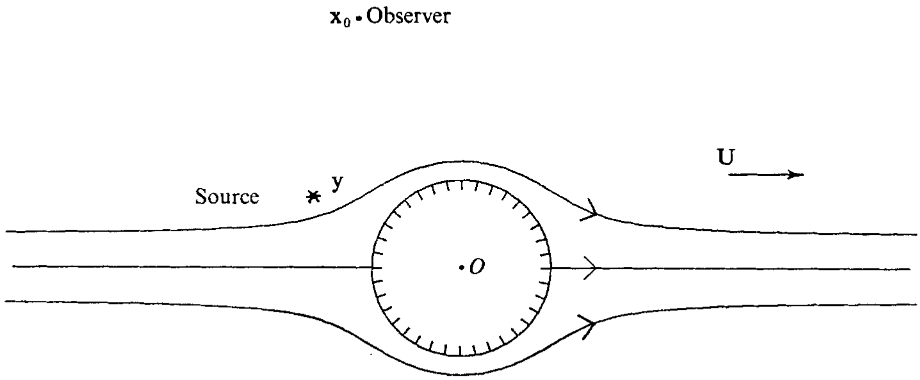

FIGURE 1. A point source is located at $\mathbf{y}$ near a compact cylinder in the presence of a steady irrotational streaming flow. It is required to determine the acoustic field at the far-field point $\mathbf{x}_0$.

Observe also that we have shown that (2.3) is valid for arbitrary incompressible $\mathbf{U}$ provided that the wavelength is small enough. This implies that in geometrical acoustics a ray path is reversible provided that the mean flow is reversed – a result which is implicit in the work of Blokhintsev (1946).

3. Scattering of sound from a point source by a rigid compact cylinder

Our first application of the reverse-flow theorem is to the problem depicted in figure 1. An acoustic point source of the type considered in §2 [equation (2.4)] is located at the point $\mathbf{y}$ near to a rigid circular cylinder of radius a whose axis lies along the x_3 axis, where $\mathbf{x} = (x_1, x_2, x_3)$. It is required to determine the acoustic potential at the far-field point $\mathbf{x}_0$, say, in the presence of a steady irrotational streaming flow about the cylinder with velocity $(U,\ 0,\ 0)$ far from the cylinder. By the reverse-flow theorem the potential at $\mathbf{x}_0$ is the same as the potential at $\mathbf{y}$ when the same source is placed at $\mathbf{x}_0$ and the mean flow is reversed. Note that, although the problem is posed in terms of a monopole source (2.4), the solution for more general source types (dipole, quadrupole, etc.) may be obtained by appropriate differentiation with respect to the source position vector $\mathbf{y}$.

We assume that the wavelength of the sound greatly exceeds the radius of the cylinder, and that the source at $\mathbf{y}$ is located well within a wavelength of the cylinder. This implies that in the vicinity of the cylinder the flow is essentially incompressible, so that in the reciprocal problem it is necessary to determine only the incompressible approximation to the potential at $\mathbf{y}$.

In the absence of the scattering cylinder the point source at $\mathbf{x}_0$ in the reciprocal problem gives rise to an *incident* potential ϕ_i, say, which satisfies (2.5a) with $\mathbf{x}_A = \mathbf{x}_0$ and $\mathbf{U} = (-U,\ 0,\ 0)$. Without loss of generality assume that the point $\mathbf{y}$ in figure 1 lies in the x_1, x_2 plane. Then provided that $M^2 \ll 1$ the incident potential at $\mathbf{y}$ is given by

$$\phi_i = \frac{1}{4\pi c^2|\mathbf{y}-\mathbf{x}_0|}\exp\left\{-i\omega\left[t-\frac{|\mathbf{y}-\mathbf{x}_0|}{c}-\frac{\mathbf{M}.(\mathbf{y}-\mathbf{x}_0)}{c}\right]\right\}, \tag{3.1}$$

where $\mathbf{M} = (U/c, 0, 0)$ (Morse & Ingard 1968, p. 717). Thus if $|\mathbf{x}_0| \gg |\mathbf{y}|$, and noting that the compactness parameter $\omega|\mathbf{y}|/c \ll 1$, we have approximately

$$\phi_i = \frac{1}{4\pi c^2|\mathbf{x}_0|}\left\{1+i\frac{\omega}{c}\mathbf{y}.\left(\mathbf{M}-\frac{\mathbf{x}_0}{|\mathbf{x}_0|}\right)+\ldots\right\}\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}_0|}{c}+\frac{\mathbf{M}.\mathbf{x}_0}{c}\right)\right\}. \quad (3.2)$$

The scattered potential ϕ_s must be such that when it is combined with (3.2) the total potential satisfies the condition of zero normal velocity on the surface of the cylinder. To the lowest-order approximation we have

$$\phi_s = \frac{1}{4\pi c^2|\mathbf{x}_0|}\frac{i\omega}{c}\mathbf{y}.\left(\mathbf{M}-\frac{\mathbf{x}_0}{|\mathbf{x}_0|}\right)\frac{a^2}{|\mathbf{y}|^2}\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}_0|}{c}+\frac{\mathbf{M}.\mathbf{x}_0}{c}\right)\right\}. \quad (3.3)$$

By the reverse-flow theorem this is also the scattered potential at $\mathbf{x}_0$ due to the source at $\mathbf{y}$ in the problem of figure 1. The scattered pressure perturbation at a general far-field point $\mathbf{x}$ is obtained by means of (2.13), and we find

$$p_s(\mathbf{x}) = -\frac{\omega\rho_0 a^2}{4\pi c^3|\mathbf{x}||\mathbf{y}|^2}\mathbf{y}.\left(\mathbf{M}-\frac{\mathbf{x}}{|\mathbf{x}|}\right)\left(1-\frac{\mathbf{M}.\mathbf{x}}{|\mathbf{x}|}\right)\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{M}.\mathbf{x}}{c}\right)\right\}, \quad (3.4)$$

correct to order M.

To interpret this result in the context of aerodynamic noise theory we now superpose on the whole system of figure 1 a uniform velocity $(-U, 0, 0)$. This gives the case of a point source fixed relative to a cylinder which is itself in uniform motion at speed U along the $-x_1$ axis in fluid at rest at infinity. It is now appropriate to express the scattered pressure (3.4) in terms of the position of the observer relative to the cylinder at the *time of emission* of the sound, rather than in terms of $\mathbf{x}$, which is the relative position on reception. Let the vector $\mathbf{R}$ represent this retarded relative position; then for $M^2 \ll 1$,

$$\left.\begin{aligned}\mathbf{x} &= \mathbf{R}+\mathbf{M}R,\\ \mathbf{x}/|\mathbf{x}| &\simeq +\frac{\mathbf{R}}{R}\left(1+\frac{\mathbf{M}.\mathbf{R}}{R}\right)^{-1}.\end{aligned}\right\} \quad (3.5)$$

Thus if Θ is the angle between $\mathbf{y}$ and $\mathbf{R}$ (see figure 2), and if θ is the angle between $-\mathbf{M}$ and $\mathbf{R}$, equation (3.4) becomes

$$p_s \simeq \frac{\omega\rho_0}{4\pi c^2}\left(\frac{\omega a}{c}\right)\left(\frac{a}{|\mathbf{y}|}\right)\frac{\cos\Theta}{(1-M\cos\theta)^3}\frac{\exp\{-i\omega(t-R/c)\}}{R}. \quad (3.6)$$

It is now clear that the scattered field is due to a dipole whose axis is along the $\Theta = 0$ direction, i.e., along the radius vector normal to the cylinder which passes through the source position. However, (3.6) also reveals that the amplification of the pressure field caused by the translation of the source–cylinder combination at speed U is proportional to the Doppler factor $(1-M\cos\theta)^{-1}$ to the power *three*. This dependence on the Doppler factor, which has also been obtained by Crighton (1974, private communication), who considered the analogous problem of a point-source–sphere combination by means of the method of matched asymptotic expansions, indicates that the combination of a source and a compact rigid scattering body in motion is *not* precisely equivalent to a point dipole.

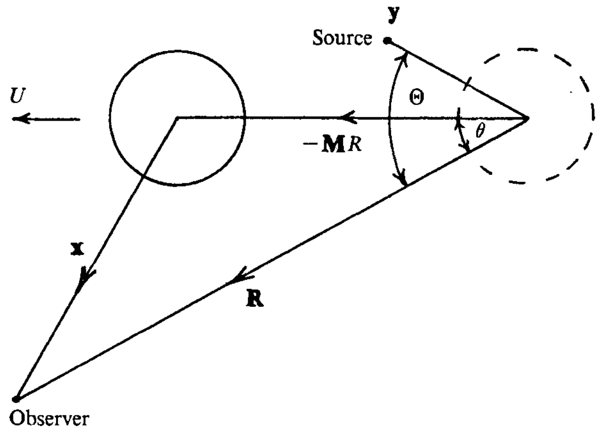

FIGURE 2. Illustrating the relation between the co-ordinate systems **x** and **R**. For simplicity **x**, **y** and **R** are shown in the same 'flyover' plane.

Indeed the acoustic pressure field of an ideal dipole in motion is amplified by only two powers of the Doppler factor. A similar result has also been obtained by Ffowcs Williams & Lovely (1974), who showed that the pressure field radiated by a compact section of a plane boundary in the presence of a uniform steady flow is amplified by three rather than two powers of the Doppler factor.

4. Scattering by an arbitrary compact body in a steady flow

The above procedure which was applied to obtain the scattered potential due to a point source located near a compact cylinder can be applied equally well to arbitrary compact source–body interaction problems in the presence of a low Mach number steady flow. For example in the case of a sphere of radius a whose centre is at the origin, it is easily deduced that the potential at a far-field point $\mathbf{x}$ due to a source

$$e^{-i\omega t}\,\delta(\mathbf{x}-\mathbf{y})$$

at $\mathbf{y}$ has the form

$$\phi(\mathbf{x},\mathbf{y},t,\omega) = \frac{1}{4\pi c^2|\mathbf{x}|}\left\{1+i\frac{\omega}{c}\mathbf{y}\,.\left(\mathbf{M}-\frac{\mathbf{x}}{|\mathbf{x}|}\right)\left(1+\frac{a^3}{2|\mathbf{y}|^3}\right)+\ldots\right\}\exp\left\{-i\omega\left[t-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{M}\,.\,\mathbf{x}}{c}\right]\right\}, \tag{4.1}$$

a formula which is valid in the compact limit $\omega|\mathbf{y}|/c \ll 1$.

In practical applications of this type of theory, however, the potential due to a stationary harmonic point source is generally not of particular interest. This is especially so in situations in which sound is generated through the unsteady convection of an acoustically silent source distribution (such as an entropy inhomogeneity) past a solid immersed in the fluid.

To treat such problems with any degree of generality it is necessary to know the Green's function of the problem $G(\mathbf{x},\mathbf{y};t,\tau)$, say, which is the solution of

$$(\partial/\partial t+\mathbf{U}\,.\,\partial/\partial\mathbf{x})^2 G - c^2\nabla^2 G = \delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau) \tag{4.2}$$

whose normal derivative vanishes on the rigid surfaces and which satisfies the radiation condition at infinity.

In low Mach number flows of the type considered here, the frequency of the acoustic bremsstrahlung generated by an interaction of the above type is determined by the ratio U/L, where U is a characteristic flow speed and the length L is a scale characteristic of the body. The corresponding acoustic wavelength is of order L/M, which greatly exceeds L for sufficiently small M. This implies that in such problems the body generally acts as a *compact* scatterer, and that therefore a knowledge of that approximation to the Green's function $G(\mathbf{x}, \mathbf{y}; t, \tau)$ appropriate to low frequency source distributions should provide an adequate mathematical description of the radiated sound.

If the exact Green's function is known, then in such compact situations its *Fourier time transform* can be developed into an asymptotic expansion in powers of the compactness parameter $\omega L/c$. If all of the frequency components of the actual source distribution are small, then only the first few terms of this expansion will be important in the determination of the acoustic radiation. Thus, for example, (4.1) may be regarded as the first *two* terms in such an expansion in the case of the source–sphere interaction problem. The corresponding approximation to the exact Green's function is obtained by multiplying (4.1) by $(2\pi)^{-1}\exp(i\omega\tau)$ and integrating over all ω. This gives

$$G(\mathbf{x}, \mathbf{y}; t, \tau) \simeq \frac{1}{4\pi c^2|\mathbf{x}|}\left\{\delta\left[t-\tau-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{M}.\mathbf{x}}{c}\right]\right.$$
$$\left.-\frac{\mathbf{y}}{c}.\left(\mathbf{M}-\frac{\mathbf{x}}{|\mathbf{x}|}\right)\left(1+\frac{a^3}{2|\mathbf{y}|^3}\right)\delta'\left[t-\tau-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{M}.\mathbf{x}}{c}\right]\right\} \qquad (4.3)$$

as the first non-trivial approximation to the exact Green's function of the problem.

To interpret this result we recall that, if the complete asymptotic expansion for the potential, approximated above by (4.1), had been used, the Green's function would have assumed the form

$$G(\mathbf{x}, \mathbf{y}; t, \tau) = \frac{1}{4\pi c^2|\mathbf{x}|}\sum_n f_n\left(\frac{\mathbf{y}}{L}\right)\left(\frac{L}{c}\right)^n \delta^{(n)}\left[t-\tau-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{M}.\mathbf{x}}{c}\right]. \qquad (4.4)$$

When this is convoluted with the actual source distribution to give the radiated sound, the neglect of terms with $n \geqslant 2$ implies that *time derivatives* of that distribution higher than the first can be neglected.

If we restrict ourselves to situations in which (4.3) gives an adequate description of the scattered radiation, then the transformation

$$\mathbf{Y} = \mathbf{y}(1+a^3/2|\mathbf{y}|^3), \qquad (4.5)$$

in which deviations from the approximation $\mathbf{Y} = \mathbf{y}$ can be regarded as a local distortion of 'free space' caused by the presence of the sphere in the flow, enables the following elegant approximation to (4.3), valid when $|\mathbf{x}| \gg |\mathbf{y}|$, to be adopted:

$$G(\mathbf{x}, \mathbf{Y}; t, \tau) = \frac{1}{4\pi c^2|\mathbf{x}-\mathbf{Y}|}\,\delta\left\{t-\tau-\frac{|\mathbf{x}-\mathbf{Y}|}{c}+\frac{\mathbf{M}.(\mathbf{x}-\mathbf{Y})}{c}\right\}. \qquad (4.6)$$

Since $|\mathbf{x}| \gg a$ we can to the same approximation replace $\mathbf{x}$ in this formula by

$$\mathbf{X} = \mathbf{x}(1+a^3/2|\mathbf{x}|^3).$$

But when this is done the resulting expression is seen to be symmetric in $\mathbf{X}$ and $\mathbf{Y}$ under change of sign of $\mathbf{M}$. In other words, by the reverse-flow reciprocal theorem

$$G(\mathbf{X}-\mathbf{Y};t-\tau) = \frac{1}{4\pi c^2|\mathbf{X}-\mathbf{Y}|}\,\delta\left\{t-\tau-\frac{|\mathbf{X}-\mathbf{Y}|}{c}+\frac{\mathbf{M}.(\mathbf{X}-\mathbf{Y})}{c}\right\} \tag{4.7}$$

provides a description of the generation of sound for an *arbitrary* source position $\mathbf{Y}$ and observation point $\mathbf{X}$ provided only that either $|\mathbf{X}| \gg a$ or $|\mathbf{Y}| \gg a$.

The following points should be noted. When $|\mathbf{X}|$, $|\mathbf{Y}| \gg a$, equation (4.7) reduces to the free-space Green's function. That is *Rayleigh scattering* of sound from a *distant* source by the sphere (in which the scattered potential is $O(\omega a/c)^2$) is neglected. In the present approximation, therefore, the presence of the sphere only modifies the sound field when it is located in the near field of the source, or when the observer is within a wavelength of the sphere.

The above results for a sphere can easily be generalized to an arbitrary compact fixed solid in the flow. To do this we note that y_i $(i = 1, 2, 3)$ is the potential of a uniform flow at unit speed in the i direction. Let the perturbation of the incompressible flow potential produced by the presence of the solid body be denoted by $\boldsymbol{\varphi}_i^*(\mathbf{y})$, where $\boldsymbol{\varphi}_i^* \simeq O(1/|\mathbf{y}|^2)$ when $|\mathbf{y}|$ is large, then we define the co-ordinate transformation analogous to (4.5) by

$$\mathbf{Y} = \mathbf{y}+\boldsymbol{\varphi}^*(\mathbf{y}), \tag{4.8}$$

after which the Green's function has the form given in (4.7).

When several compact scatterers are present in the flow (but 'diffusely' spread on a wavelength scale), $\boldsymbol{\varphi}^*$ may be regarded as defining the perturbation of a uniform flow about the whole system of bodies. The transformation (4.8) is the identity transformation far from the individual scattering bodies, but near a body $\boldsymbol{\varphi}^*$ is large, and the body can be imagined to distort locally the geometry of the three-dimensional Euclidean space occupied by the medium.

5. Acoustic bremsstrahlung in a low Mach number duct flow

As an illustration and extension of the ideas introduced in the previous section we now determine the acoustic radiation emitted when a point source of *constant* strength q is convected in a low Mach number irrotational flow through a neck in a hard-walled duct (see figure 3). When the source is located far from the neck, in the region of uniform flow, no acoustic pressure fluctuations are developed. The unsteadiness of the source motion in the vicinity of the neck, however, is accompanied by the emission of sound waves of characteristic frequency U/L, where L is the length scale of the necking and U is the characteristic convection speed. A typical wavelength $\sim L/M$ is therefore large in comparison with the neck scale L, and the perturbed flow in the neck may consequently be treated as if it were incompressible.

To solve this problem we first determine the lowest-order terms in the low

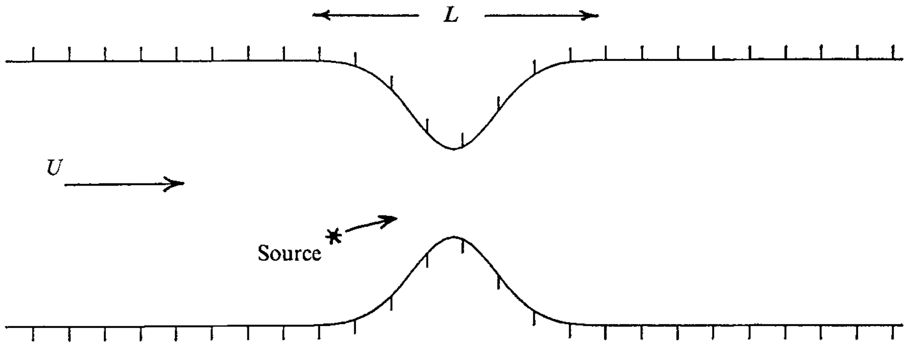

FIGURE 3. A point source of constant strength q is convected by a steady irrotational flow through a neck in a duct.

frequency expansion of the Green's function for an observation point $\mathbf{x}$ far from the neck. Thus we consider the solution of the equation

$$(\partial/\partial t+\mathbf{U}\,.\,\partial/\partial\mathbf{x})^2\phi-c^2\nabla^2\phi = e^{-i\omega t}\,\delta(\mathbf{x}-\mathbf{y}), \tag{5.1}$$

where the source is located at a point $\mathbf{y}$ in the region of the neck. As before the analysis is conducted in terms of a reciprocal problem

$$(\partial/\partial t-\mathbf{U}\,.\,\partial/\partial\mathbf{x})^2\bar{\phi}-c^2\nabla^2\bar{\phi} = e^{-i\omega t}\,\delta(\mathbf{x}-\bar{\mathbf{x}}), \tag{5.2}$$

in which the irrotational convection velocity is reversed at all points of the flow and the source is located at $\bar{\mathbf{x}}$, $|\bar{\mathbf{x}}| \gg |\mathbf{y}|$. Then by the reverse-flow theorem

$$\phi(\bar{\mathbf{x}}) = \bar{\phi}(\mathbf{y}). \tag{5.3}$$

Now for the low frequencies contemplated in this problem the disturbance generated by the distant source in (5.2) will develop into a plane propagating incident wave on reaching the neck. Introduce a system of co-ordinates in which the $+x_1$ axis is in the direction of the mean flow in the duct, with the origin located in the region of the neck. Then in the case in which $\bar{x}_1$ is positive, the potential of the incident wave in the reciprocal problem is given by

$$\bar{\phi}_I \equiv \bar{\phi}_I\left(t+\frac{x_1}{c(1+M)}\right) = \frac{i}{2\omega cA}\exp\left\{\frac{i\omega\bar{x}_1}{c(1+M)}-i\omega\left[t+\frac{x_1}{c(1+M)}\right]\right\}, \tag{5.4}$$

where A is the asymptotic cross-sectional area of the duct, $M = U/c$, and U is the uniform flow speed far from the neck. At the neck a reflected wave $\bar{\phi}_R$ and a transmitted wave $\bar{\phi}_T$ are generated. Far from the neck these have the following respective functional forms:

$$\left.\begin{aligned}\bar{\phi}_R &\equiv \bar{\phi}_R(t-x_1/c(1-M)) \quad (x_1>0),\\ \bar{\phi}_T &\equiv \bar{\phi}_T(t+x_1/c(1+M)) \quad (x_1<0).\end{aligned}\right\} \tag{5.5}$$

In the approximation in which terms of order $(\omega L/c)^2$ and higher are neglected in the low frequency expansion of the Green's function, the flow in the neighbourhood of the duct neck is incompressible, and the potential there is given by

$$\bar{\phi} = \psi_0(t)+\psi_1(t)\,\phi^*(\mathbf{x}), \tag{5.6}$$

where the second term on the right is $O(\omega L/c)$ relative to the first. The potential function $\phi^*(\mathbf{x})$ describes steady irrotational flow through the neck, and is assumed to be known. It is normalized in such a manner that

$$\bar{\phi} \to \psi_0(t) + \psi_1(t)\,x_1 \quad \text{as} \quad x_1 \to \pm\infty. \tag{5.7}$$

This asymptotic expression must match the corresponding terms in the expansion of the acoustic wave field for small retarded times $x_1/c(1\pm M)$, and this implies that

$$\left.\begin{aligned} \bar{\phi}_T(t) &= \psi_0(t) = \bar{\phi}_I(t) + \bar{\phi}_R(t), \\ \frac{1}{c(1+M)}\frac{\partial \bar{\phi}_T(t)}{\partial t} &= \psi_1(t) = \frac{1}{c(1+M)}\frac{\partial \bar{\phi}_I(t)}{\partial t} - \frac{1}{c(1-M)}\frac{\partial \bar{\phi}_R(t)}{\partial t}. \end{aligned}\right\} \tag{5.8}$$

It follows that, correct to order M,

$$\bar{\phi}(\mathbf{x},t) = \bar{\phi}_I(t) + \frac{\phi^*(\mathbf{x})}{c(1+M)}\frac{\partial \bar{\phi}_I(t)}{\partial t}, \tag{5.9}$$

i.e., using (5.4),

$$\bar{\phi}(\mathbf{x},t) = \frac{i}{2\omega c A}\left[1 - \frac{i\omega\phi^*(\mathbf{x})}{c(1+M)}\right]\exp\left\{-i\omega\left(t - \frac{\bar{x}_1}{c(1+M)}\right)\right\}. \tag{5.10}$$

By the reverse-flow theorem this is also the potential $\phi(\bar{\mathbf{x}})$ at $\bar{\mathbf{x}}$ due to a harmonic point source located at $\mathbf{x}$. The corresponding approximation to the Green's function $G(\mathbf{x}, \mathbf{y};\, t, \tau)$ due to a source $\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$ located at $\mathbf{y}$ in the original problem is obtained by Fourier integration with respect to ω, the causality condition being satisfied by indenting the path of integration to pass above the singularity at $\omega = 0$. This gives

$$G(\mathbf{x},\mathbf{y};t,\tau) = \frac{1}{2cA}H\left(t-\tau-\frac{x_1}{c(1+M)}\right) + \frac{1}{2c^2A}\frac{\phi^*(\mathbf{y})}{(1+M)}\,\delta\left(t-\tau-\frac{x_1}{c(1+M)}\right), \tag{5.11}$$

where H is the Heaviside unit function.

As in §4 a co-ordinate transformation can be introduced and is defined here by

$$Y_1 = \phi^*(\mathbf{y}), \quad Y_2 = y_2, \quad Y_3 = y_3, \tag{5.12}$$

in which $Y_1 \to y_1$ when $|\mathbf{y}|$ is large. Then the terms in (5.11) can be approximated further by

$$G(\mathbf{x},\mathbf{Y};t,\tau) = \frac{1}{2cA}H\left(t-\tau-\frac{(x_1-Y_1)}{c(1+M)}\right). \tag{5.13}$$

This expression has been deduced on the assumption that $x_1 \gg L > 0$. A similar calculation for points x_1 upstream of the neck reveals that G can be expressed in a form valid for all $|x_1| \gg L$, viz.

$$G(\mathbf{x},\mathbf{Y};t,\tau) = \frac{1}{2cA}H\left(t-\tau-\frac{|x_1-Y_1|}{c[1+M\,\mathrm{sgn}\,(x_1-Y_1)]}\right). \tag{5.14}$$

Again, as in §4 this result can be generalized by replacing $\mathbf{x}$ by $\mathbf{X}$, and this gives our final version of the low frequency approximation to the Green's function:

$$G(\mathbf{X}-\mathbf{Y};t-\tau) = \frac{1}{2cA}H\left(t-\tau-\frac{|X_1-Y_1|}{c[1+M\,\mathrm{sgn}\,(X_1-Y_1)]}\right). \tag{5.15}$$

This generalization is valid for an *arbitrary* source location $\mathbf{Y}$ *and* observation point $\mathbf{X}$ provided only that at least one of these points is at least a wavelength or so from the neck.

We are now in a position to solve the problem posed at the beginning of this section. We consider a point source of constant strength q convected along the duct at the locally steady mean flow velocity $\mathbf{U}(\mathbf{x})$. Let $\mathbf{x}_0(t)$ denote the position of the source at time t. The equation for the potential of the acoustic field is then

$$(\partial/\partial t + \mathbf{U}.\partial/\partial\mathbf{x})^2\phi - c^2\nabla^2\phi = -c^2 q\delta[\mathbf{x}-\mathbf{x}_0(t)]. \tag{5.16}$$

Using the Green's function (5.15), with the observation point $\mathbf{x} \simeq \mathbf{X}$ far from the neck of the duct, we have

$$\begin{aligned}\phi &\simeq -\frac{cq}{2A}\int \delta[\mathbf{y}-\mathbf{x}_0(\tau)]\,H\left(t-\tau-\frac{|x_1-Y_1|}{c(1+M\operatorname{sgn}x_1)}\right)d^3\mathbf{y}\,d\tau \\ &= -\frac{cq}{2A}\int H\left(t-\tau-\frac{|x_1-\phi^*(\mathbf{x}_0(\tau)|}{c(1+M\operatorname{sgn}x_1)}\right)d\tau. \end{aligned}\tag{5.17}$$

Now the parameter of physical significance is the acoustic pressure perturbation p, given in terms of the perturbation potential ϕ by (2.13). Applying this to the result (5.17) we have

$$p \simeq \frac{cq\rho_0}{2A}\int [1-M\operatorname{sgn}x_1]\,\delta\left(t-\tau-\frac{|x_1-\phi^*(\mathbf{x}_0(\tau)|}{c(1+M\operatorname{sgn}x_1)}\right)d\tau$$

for $M^2 \ll 1$, i.e.,

$$p \simeq \frac{cq\rho_0}{2A}\frac{1-M\operatorname{sgn}x_1}{\left[1-\dfrac{\partial\phi^*}{\partial\tau}\dfrac{\operatorname{sgn}x_1}{c(1+M\operatorname{sgn}x_1)}\right]_{\tau=T}},$$

where $T = t-|x_1|/c(1+M\operatorname{sgn}x_1)$. Since

$$\partial\phi^*/\partial\tau = \nabla\phi^*.\mathbf{U}(\mathbf{x}_0(\tau))$$

and $M^2 \ll 1$, this becomes

$$\begin{aligned} p &\simeq \frac{cq\rho_0}{2A}\left[1+\frac{\operatorname{sgn}x_1}{cU}\{\mathbf{U}[\mathbf{x}_0(T)]-U^2\}+\ldots\right] \\ &= \frac{cq\rho_0}{2A}+\operatorname{sgn}x_1\left(\frac{q}{AU}\right)\frac{\rho_0}{2}[\mathbf{U}^2[\mathbf{x}_0(T)]-U^2]. \end{aligned}\tag{5.18}$$

When the source is far from the neck $\mathbf{U}[\mathbf{x}_0(T)] = (U,0,0)$ and the second term in this expression vanishes. The first term is constant, and actually represents the constant pressure perturbation associated with a point source of constant strength convecting along a duct of uniform cross-section. Evidently the time-dependent part of (5.18) describes the radiation emitted during the passage of the source through the neck.

Since steady irrotational flow in the duct satisfies Bernoulli's equation in the form

$$p_0(\mathbf{x})+\tfrac{1}{2}\rho_0\mathbf{U}^2(\mathbf{x}) = p_\infty+\tfrac{1}{2}\rho_0 U^2, \tag{5.19}$$

where $p_0(\mathbf{x})$ is the steady pressure at $\mathbf{x}$ and p_∞ is the pressure in the uniform flow far from the neck, the acoustic pressure perturbation is also given by

$$p \sim \operatorname{sgn} x_1 \left(\frac{q}{AU}\right) \left\{p_\infty - p_0 \left[\mathbf{x}_0\left(t - \frac{|x_1|}{c(1+M \operatorname{sgn} x_1)}\right)\right]\right\}. \tag{5.20}$$

In the region of the neck $p_0(\mathbf{x}) < p_\infty$, so that, if $q > 0$, equation (5.20) reveals that the acoustic field consists of a compression wave which is radiated downstream ahead of the source, and an expansion wave radiated upstream from the neck. The strength of the radiation is directly proportional to q/AU, the ratio of the mass flux from the source to the constant mass flow down the duct, and the peak acoustic pressure level is proportional to the largest steady pressure defect of the ambient fluid along the trajectory of the source through the neck.

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